Table 2 Mean theoretical illumination profile for 31 polydispersions obeying the ULDF

Reduced angle $\pi D_{32}\theta/\lambda$	Mean illumination $I(\theta)$	Standard deviation from mean $\pi D_{32}\theta/\lambda$ as a percentage of mean value
0	1 00	0
0 484	$9\ 20 \times 10^{-1}$	7 11
1 014	$7\ 00 \times 10^{-1}$	6 76
1 631	$4\ 00 \times 10^{-1}$	5 58
2 176	$2\ 00 \times 10^{-1}$	3 83
2 647	$1\ 00 \times 10^{-1}$	1 74
2 992	$6\ 00 \times 10^{-2}$	1 41
3 540	$3\ 00 \times 10^{-2}$	4 44
3 977	$2\ 00 \times 10^{-2}$	5 78
4 696	$1\ 30 \times 10^{-2}$	4 48

0 28) Using Figs 1–3, appropriate values of a and δ were determined for use in Eq (4) Values of Q were chosen to encompass both wide and narrow distributions However, values of Q which resulted in distributions that were sufficiently narrow as to approach a monodispersion, as characterized by distinct diffraction rings occurring in the corresponding illumination profile, were not included, as they would easily be distinguished as a monodispersion in an actual experimental measurement The actual values of the parameters for the distributions chosen are given in Table 1

As in the original work,[1] it was found subsequently[4] that when D_0 was set equal to D_{32} the various illumination profiles for all the distributions in Table 1 were almost coincident for values of $I(\theta)$ to 0 01 Furthermore, the mean curve for all the cases considered was very nearly coincident with the mean curve obtained for distributions having $0\ 13 < \bar{D}/D_\infty < 0\ 28$ [1] This result naturally led to an investigation of the complete range of distributions (i e , $0\ 13 < \bar{D}/D_\infty < 0\ 8$) with a determination of the mean curve for $I(\theta)$ vs $\pi D_{32}\theta/\lambda$ and the percent standard deviation from the mean for a total of 31 different distributions (18 from Ref 1, and 13 here) These results are tabulated in Table 2 and the mean curve plotted in Fig 4

In fitting an experimentally determined illumination profile to the curve of Fig 4, an obvious choice for determining D_{32} is in the region of $I(\theta) = 8 \times 10^{-2}$, where the standard deviation is only slightly greater than 1% The close coincidence of all the theoretical illumination profiles in this region is a consequence of the occurence of discreet diffraction rings for monodispersions, the first ring occurring at $\pi D_{32}\theta/\lambda = 3\ 83$ Although no distributions selected here would allow a ringed structure in the illumination profile, some were sufficiently narrow to follow quite closely the characteristics of a true monodispersion, except in the regions close to where the rings would occur The broad distributions, on the other hand, deviate from the monodispersion illumination profile, following below it for $\pi D_{32}\theta/\lambda < 3$ and above it for $\pi D_{32}\theta/\lambda > 3$ The cumulative effect of these trends produces the small total dispersion in the region noted previously

From the results, it is concluded that a value of D_{32} may be determined from the intensity of diffractively scattered light from a polydispersion of spherical particles to a good degree of accuracy for extremely wide ranges of distributions and without any knowledge of general distribution type It is felt that this conclusion appreciably extends the usefulness of the technique for measurement of mean particle size (D_{32}), especially since it may be used together with the results from a simple optical transmission test to determine particle concentration [1]

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Skip-Impact Criteria of a Re-Entry Trajectory with Negative Lift

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Nomenclature

A = frontal area of re entry body, ft^2
C_D = drag coefficient
C_L = lift coefficient
g = gravitational acceleration, ft/sec^2
h = altitude, ft
m = mass, slugs
r = distance from center of earth to re entry body, ft
t = time, sec
V = velocity, fps
W = weight, lb
β = constant in expression for exponential atmosphere 1/22,000, ft^{-1}
Δ = W/C_DA psf
θ = flight path angle, rad
ρ = atmospheric density, slug/ft^3

Subscripts

C = circular, critical
e = entry
m = minimum
s = sea level
$*$ = critical

FOR nonlifting re-entry trajectories it is known that if a nonlifting body re-enters the atmosphere with subcircular re-entry velocity the trajectory will be of the direct-impact type, whereas for a supercircular re-entry velocity the trajectory will be either a skip- or direct-impact type, depending on the values of the re-entry angle and velocity Kornreich[1] presents an approximate method foı determining the re-entry conditions for a nonlifting body at which the trajectory changes from skip- to direct-impact type The skip-type trajectories mav be prevented by using enough negative lift to force a direct-impact trajectory This note presents an approximate method for determining the critical re-entry angle at which a trajectory switches from a skip- to a direct-impact type when given a re-entry velocity and a constant negative lift coefficient Three equations in three unknowns are presented along with an iteration procedure, which yields the value of the critical re-entıy angle The latter is found to be in good agreement with values obtained by direct numerical integration of the equations of motion

The equation of motion along the direction of flight for a vehicle with aerodynamic forces is

$$\frac{d\theta}{dt} = -\frac{\rho V C_L A}{2m} + \left(\frac{g}{V} - \frac{V}{r}\right)\cos\theta \qquad (1)$$

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Table 1 Critical re-entry angle comparison for various re-entry conditions

V_e/V_C	W/C_DA, psf	W/C_LA, psf	C_L	θ_C, rad (analytic approximation)	θ_C, rad (machine calculation)
1 2	6 25	−6 25	−0 8	0 050	0 050
1 2	11 76	−25 00	−0 2	0 058	0 058
1 2	58 82	−125 00	−0 2	0 066	0 066
1 2	121 95	−125 00	−1 0	0 066	0 067
1 2	156 25	−156 25	−0 8	0 067	0 068
1 2	294 12	−625 00	−0 2	0 074	0 074
1 7	6 25	−6 25	−0 8	0 080	0 078
1 7	11 76	−25 00	−0 2	0 093	0 089
1 7	58 82	−125 00	−0 2	0 102	0 101
1 7	156 25	−156 25	−0 8	0 104	0 102
1 7	294 12	−625 00	−0 2	0 112	0 111

Introducing an approximate exponential atmosphere and differentiating with respect to time gives

$$\rho = \rho_S \exp(-\beta h)$$

$$d\rho/dt = \beta\rho V \sin\theta$$

since

$$dh/dt = -V \sin\theta$$

Using this with Eq (1) we have

$$\frac{d\theta}{d\rho} = -\frac{C_LA}{2m\beta \sin\theta} + \frac{1}{\beta\rho}\left(\frac{g}{V^2} - \frac{1}{r}\right)\cot\theta \tag{2}$$

In a direct impact trajectory, when $V_e > V_C$, θ decreases from θ , reaches a minimum θ_m, and then increases again Therefore, at θ_m we set $d\theta/d\rho = 0$ in Eq (2):

$$-\frac{C_LA}{2m\beta} + \frac{1}{\beta\rho_*}\left(\frac{g}{V_*^2} - \frac{1}{r}\right)\cos\theta_m = 0 \tag{3}$$

For a skip trajectory, θ decreases to zero and then becomes negative At the critical point where the skip and impact trajectories coincide, we have $\theta = \theta_m = 0$ Making the assumption that $r \approx r_S$, since $h \ll r$, the density at the critical point is given by

$$\rho_* = \frac{2m}{C_LA}\left(\frac{g}{V_*^2} - \frac{1}{r_S}\right) \tag{4}$$

Equation (2) may be written as

$$\sin\theta d\theta = -\frac{C_LA}{2m\beta}\,d\rho - \left(\frac{1}{\beta r} - \frac{g}{\beta V^2}\right)\cos\theta\,\frac{d\rho}{\rho} \tag{5}$$

In order to integrate Eq (5) we make the assumption that

$$\left(\frac{1}{\beta r} - \frac{g}{\beta V^2}\right)\cos\theta = \left(\frac{1}{\beta r_S} - \frac{g}{\beta V^2}\right)\cos\theta \tag{6}$$

This assumption is valid since V and $\cos\theta$ do not change much between entry and the critical point Integrating Eq (5) between entry and the critical point ($\theta = 0$) and solving the resulting relation for $\cos\theta$,

$$\cos\theta_C = \frac{1 - (C_LA/2m\beta)(\rho_* - \rho_e)}{1 + (1/\beta)[(1/r_S) - (g/V_e^2)]\ln(\rho_*/\rho)} \tag{7}$$

where θ_C is the critical re-entry angle

Wang and Ting[2] presented the following relationship between the velocity, density, and flight path angle under similar assumptions, when the re-entry velocity is greater than circular:

$$\Delta \ln\frac{V_e}{V} = \frac{\rho}{2\beta}\,\frac{1}{-b^{1/2}} \times$$

$$\ln\left\{\frac{a - 4b + 2\sigma b + 2[-b\theta^2 + (a - 3b + \sigma b)(\sigma b - b)]^{1/2}}{2(-b)^{1/2}\theta + a - 2b}\right\} \tag{8}$$

where

$$\Delta = \frac{W}{C_DA} \qquad \sigma = \frac{\rho_e}{\rho}$$

$$a = \frac{C_LA}{W\beta}\,\rho \qquad b = \left(\frac{g}{V_e^2} - \frac{1}{r_S}\right)\frac{\cos\theta_e}{\beta}$$

At the critical point, $\theta = 0$, $\rho = \rho_*$, and $V = V_*$ Therefore

$$\ln\frac{V_e}{V_*} = \frac{\rho_* C_DA}{2W\beta}\,\frac{1}{(-b_*)^{1/2}} \times$$

$$\ln\left\{\frac{a_* - 4b_* + 2\sigma_* b_* + 2[(a_* - 3b_* + \sigma_* b_*)(\sigma_* b_* - b_*)]^{1/2}}{a_* - 2b_*}\right\} \tag{9}$$

with

$$\sigma_* = \frac{\rho_e}{\rho_*} \qquad a_* = \frac{C_LA}{W\beta}\,\rho_* \qquad b_* = \left(\frac{g}{V_*^2} - \frac{1}{r_S}\right)\frac{\cos\theta_C}{\beta}$$

Equations (4, 7, and 9) now relate ρ_*, θ_C, and V_*

Since there is not much of a change in velocity between entry and the critical point, and since the critical point only exists for supercircular re-entry speeds, the velocity at the critical point must be greater than circular velocity Therefore, the following iterative procedure may be used to determine θ_C As a first approximation, assume the velocity at the critical point V_* equal to the average of the re-entry and circular velocities Equation (4) is then used to obtain ρ_*, which in turn determines θ_C by means of Eq (7) These values are then used in Eq (9) to calculate a new value of V_* If the assumed and calculated values of V_* do not agree to the desired accuracy, use the average of these two values as a new trial value of V_* and repeat until the desired accuracy is reached The averaging is because of the oscillatory nature of the iteration In all cases investigated, very rapid convergence was observed using this method A few cases may exist where this method diverges because of the initial value of V_* This difficulty can be overcome by obtaining a new trial value by averaging the divergent trial value with either the re-entry or circular velocity, if the initial value is, respectively, too small or too large, and repeating until a convergent solution is obtained

The final entry angle obtained is the one at which the trajectory changes from skip- to impact-type, or vice versa For a given set of trajectory parameters, a skip trajectory will result if $\theta < \theta_C$, and a direct impact trajectory will result if $\theta > \theta_C$

Results obtained by this method were compared to those obtained by direct numerical integration of the equations of motion Good agreement was found for re-entry velocities of about 31,000 and 44,000 fps with values of W/C_DA between 6 and 300 psf and values of C_L between −0 2 and −1.0, all for an atmospheric entry altitude of 400,000 ft The results of the machine and analytic calculations are shown in Table 1

These results show that the important body parameter is the loading W/C_LA Comparison of the two cases with reentry velocity 1 2 times circular velocity and W/C_DA = 58 82 and 121 95 psf, shows that, since W/C_LA is the same for both, the critical angles are the same even though the values of C_L differ appreciably (−0 2 and −1 0)

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Technical Comments

Comment on "An Approximate Solution for Laminar Boundary Layer Flow"

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IN a recent paper[1] Kosson presented an approximate solution for two-dimensional, incompressible, laminar boundary-layer flow with an arbitrary pressure gradient As an example of his method Kosson considered the external flow $U = 2U_\infty \sin(x/R)$ past a circular cylinder of radius R Kosson compared the results obtained by his method with a series solution given by Ulrich[2] using, presumably, the more accurate values for the coefficients of the terms of the series obtained by Tifford [3] However, an exact numerical solution for the flow has been obtained by Terrill,[4] and the results given by Kosson for nondimensional skin friction, displacement thickness, and momentum thickness are compared with Terrill's results in Table 1

Kosson points out that "a higher-order polynomial is required in order for the series expansion method to be valid " and that part of the discrepancy in the region of decelerating flow "may be attributed to errors in the series solution " The reason for the slow convergence of the series expansion near separation is that, for this external flow, there is a singularity in the laminar boundary-layer equations at the separation point (discussed in Ref 4) Nevertheless, it can be seen from Table 1 that there is not a great difference between the series and the exact solutions at an angle of 100° from the leading edge Near separation the skin friction behaves like $\xi^{1/2}$, where ξ is the distance from separation, and so decreases very rapidly as the separation point is approached It is not surprising that a series method, for which the skin friction is almost certain to fall less rapidly than for an exact numerical solution (because of the singularity), predicts separation later than 104 45° It is more surprising that Kosson's solution gives separation before the correct value; this indicates that Kosson's values for the skin friction are much less than the true values near separation and is confirmed by the results at η = 100° However, there appears to be good agreement between his results and the exact results for the displacement and momentum thicknesses at the separation point

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Table 1 Results for the external flow $U = 2U_\infty \sin(x/R)$

	$\frac{\tau_0}{\rho U^2_\infty}\left(\frac{U_\infty R}{\nu}\right)^{1/2}$			$\delta^*\left(\frac{U_\infty}{\nu R}\right)^{1/2}$			$\theta\left(\frac{U_\infty}{\nu R}\right)^{1/2}$		
η°	Kosson	Series	Exact	Kosson	Series	Exact	Kosson	Series	Exact
0	0	0	0	0 456	0 46	0 458	0 203	0 21	0 207
30	1 62	1 64	1 64	0 481	0 49	0 485	0 212	0 22	0 218
60	2 22	2 26	2 25	0 580	0 59	0 585	0 250	0 26	0 26
90	1 26	1 35	1 35	0 918	0 89	0 89	0 357	0 35	0 36
100	0 34	0 71	0 64	1 372	1 12	1 22	0 443	0 40	0 44
102 45	0			1 758			0 472		
104 45			0			1 704			0 484
108 8		0			1 45			0 40	

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